XXVI. *A Specimen of a new Method of comparing Curvilineal Areas; by which many ſuch Areas may be compared as have not yet appeared to be comparable by any other Method.* By John Landen, *F. R. S.*

Read June 9, 1768.

WHEN a body in motion is continually acted upon by a variable force, the ſpace it has paſſed over at the end of any given time, it is well known, will be expreſſed by the area of the curve, whoſe ordinate expreſſes the velocity of the body, whilſt the time it has been in motion is expreſſed by the correſponding abſciſſa. Therefore the facilitating the computation of curvilineal areas will manifeſtly contribute to the improvement of the doctrine of motion. Which doctrine being a branch of philoſophy of no ſmall importance, ſuch improvement will not, I am perſuaded, be looked upon as a trifling ſpeculation, by the Royal Society: to whom, therefore, I do myſelf the honour of communicating this ſpecimen of a new and ready method of computing ſuch areas, by means of a given area; preſuming that what is here written will not be deemed undeſerving of a place in the Philoſophical Tranſactions.

I.

Geometricians have found, that, if A be put to denote the *whole* area of the curve, whoſe abſciſſa is x, and ordinate $\overline{1-x^2}|^{p-1} \times x^{2r-1}$, the

the *whole* area of the curve, whose abscissa is x, and ordinate $\overline{1-x^2}|^{p-1} \times x^{2n+2r-1}$ will be $= \frac{r.r+1.r+2\ (n)}{p+r.p+r+1.p+r+2\ (n)} \times A$; n being any positive integer, and p and r any positive numbers, whole or fractional.

II.

By the preceding article, the *whole* area, when the ordinate is $\overline{1+x}|^{p-1} \times x^{2r+2z-1}$ is $= \frac{z.z+1\ (r)}{p+z.p+z+1\ (r)} \times \frac{1.2\ (z-1)}{p.p+1\ (z)} \times \frac{1}{2}$; the *whole* area, when the ordinate is $\overline{1-x}|^{p-1} \times x^{2z-1}$, being $= \frac{1.2\ (z-1)}{p.p+1\ (z)} \times \frac{1}{2}$.

Likewise, by the same article, the same *whole* area is $= \frac{r.r+1.r+2\ (z)}{p+r.p+r+1.p+r+2\ (z)} \times A$. Therefore this last expression is $= \frac{z.z+1\ (r)}{p+z.p+z+1\ (r)} \times \frac{1.2\ (z-1)}{p.p+1\ (z)} \times \frac{1}{2}$. From which equation, p and r being positive, as before observed, A, the *whole* area of the curve, whose ordinate is $\overline{1-x^2}|^{p-1} \times x^{2r-1}$, is found equal to $\frac{1.2.3\ (r+z-1) \times p+r.p+r+1\ (z)}{p.p+1.p+2\ (r+z) \times r.r+1\ (z)} \times \frac{1}{2}$, being any number whatever.

Consequently, supposing z infinite, we find A = the ultimate value, or limit of $\frac{1.2.3\ (z) \times p+r.p+r+1\ (z)}{p.p+1.p+2\ (z) \times r.r+1\ (z)} \times \frac{1}{2z}$.

Having thus obtained a general expression for the *whole* area of any curve, whose ordinate is expressed by $\overline{1-x^2}|^{p-1} \times x^{2r-1}$, and that

expression

expreſſion for ſuch area conſiſting of an infinite number of factors multiplied together; to render the ſame uſeful in practice, ſome theorems are requiſite for aſcertaining the limits of ſuch products. The theorems which I have hitherto been able to inveſtigate ſuitable tohat purpo ſe, I ſhall give in the next two articles.

III.

The limit of $\overline{1-m^2}\times\overline{2^2-m^2}\times\overline{3^2-m^2}\ (z)\times\frac{N^{2z}}{z^{2z+1}}$ is $=\frac{2}{m}\times$ ſine of mS; N being the number whoſe hyp. log. is 1, and S the ſemi-periphery of the circle, whoſe radius is 1.

Whencc, by taking m equal o, we find the limit of $1^2.\ 2^2.\ 3^2$ $(z)\times\frac{N^{2z}}{z^{2z+1}}=2S$,

IV.

The limit of $\overline{dz+a}\times\overline{dz+a+d}\times\overline{dz+a+2d}\times\frac{N^z}{2^{2z}d^z z^z}$ is $=2^{\frac{a}{d}-\frac{1}{2}}$.

Hence, $\overline{z+1}.\overline{z+2}.\overline{z+3}\ (z)$ being $=1.3,5\ (z)\times2^z$, it appears, that the limit of $1.3.5\ (z)\times\frac{N^z}{2^z z^z}$ is $=2^{\frac{1}{2}}$.

I ſhall now give ſome examples, to ſhew the uſe of the above articles.

Writing

V.

Writing A, B, C, D, and E, for the *whole* areas of the curves, whose ordinates are $\frac{x^{\frac{2}{3}}}{\overline{1-x^2}|^{\frac{1}{2}}}$, $\frac{x^{\frac{1}{3}}}{\overline{1-x^2}|^{\frac{1}{2}}}$, $\frac{1}{\overline{1-x^2}|^{\frac{1}{2}}}$, $\frac{x^{-\frac{1}{3}}}{\overline{1-x^2}|^{\frac{1}{2}}}$, and $\frac{x^{-\frac{2}{3}}}{\overline{1-x^2}|^{\frac{1}{2}}}$, respectively; we have, by Art. II.

$$A = \text{the limit of } \frac{1.2.3\,(z) \times 4.\ 7.10\,(z)}{1.3.5\,(z) \times 5.11.17\,(z)} \times \frac{2^{2z-1}}{z};$$

$$B = \text{the limit of } \frac{1.2.3\,(z) \times 7.13.19\,(z)}{1.3.5\,(z) \times 2.\ 5.\ 8\,(z)} \times \frac{1}{2z};$$

$$C = \text{the limit of } \frac{1^2.\ 2^2.\ 3^2\,(z)}{1^2.\ 3^2.\ 5^2\,(z)} \times \frac{2^{2z-1}}{z} = \left\{\begin{array}{l}\text{the area of the semi-circle,}\\ \text{whose radius is } 1;\end{array}\right.$$

$$D = \text{the limit of } \frac{1.2.3\,(z) \times 5.11.17\,(z)}{1.3.5\,(z) \times 1.\ 4.\ 7\,(z)} \times \frac{1}{2z};$$

$$E = \text{the limit of } \frac{1.2.3\,(z) \times 2.\ 5.\ 8\,(z)}{1.3.5\,(z) \times 1.\ 7.13\,(z)} \times \frac{2^{2z-1}}{z}.$$

Now it appears by the above equations, that $\frac{A}{B}$ is = the limit of $\frac{2.4.\ 5.\ 7.\ 8.10\,(2z)}{5.7.11.13.17.19\,(2z)} \times 2^{2z}$; which by Art. III. is $= \frac{6 \times \text{sine } 60^\circ}{12 \times \text{sine } 30^\circ} = \frac{3^{\frac{1}{2}}}{2}$.

Therefore A is $= \frac{3^{\frac{1}{2}} B}{2}$.

It appears alſo, that $\frac{B\times D}{C}$ is $=$ the limit of $\frac{5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\ (2z)}{2\cdot 4\cdot\ 5\cdot\ 7\cdot\ 8\cdot 10\ (2z)}\times\frac{3}{2^{2z+1}}$; which by Art. III. is $= 3^{\frac{1}{2}}$.

Therefore D is $= \frac{3^{\frac{1}{2}}C}{B}$.

It likewiſe appears, that

B $\times$ E is $=$ the limit of $\frac{1^2\cdot 2^2\cdot 3^2\ (z)}{1^2\cdot 3^2\cdot 5^2\ (z)}\times\frac{3\cdot 2^{2z-1}}{z} = 3\,C$.

Therefore E $= \frac{3\,C}{B}$.

VI.

Writing F, G, H, I, and K, for the *whole* areas of the curves, whoſe ordinates are $\frac{x^{\frac{2}{3}}}{\overline{1-x^2}|^{\frac{1}{3}}}$, $\frac{x^{\frac{1}{3}}}{\overline{1-x^2}|^{\frac{1}{3}}}$, $\frac{1}{\overline{1-x^2}|^{\frac{1}{3}}}$, $\frac{x^{-\frac{1}{3}}}{\overline{1-x^2}|^{\frac{1}{3}}}$, and $\frac{x^{-\frac{2}{3}}}{\overline{1-x^2}|^{\frac{1}{3}}}$, reſpectively, we have, by Art. II.

F $=$ the limit of $\frac{1\cdot 2\cdot 3\ (z)\times 9\cdot 15\cdot 21\ (z)}{2\cdot 5\cdot 8\ (z)\times 5\cdot 11\cdot 17\ (z)}\times\frac{3^z}{2z}$;

G $=$ the limit of $\frac{1\cdot 2\cdot 3\ (z)\times 8\cdot 14\cdot 20\ (z)}{2^2\cdot 5^2\cdot 8^2\ (z)}\times\frac{3^z}{2^{z+1}\times z}$;

H $=$ the limit of $\frac{1\cdot 2\cdot 3\ (z)\times 7\cdot 13\cdot 19\ (z)}{2\cdot 5\cdot 8\ (z)\times 1\cdot\ 3\cdot\ 5\ (z)}\times\frac{1}{2z} = B$;

I $=$ the

I = the limit of $\frac{1^2.\ 2^2.\ 3^2\ (z)}{2.5.8\ (z)\ \times\ 1.4.7\ (z)} \times \frac{3^{2z}}{2z}$;

K = the limit of $\frac{1.2.3\ (z) \times 5.11.17\ (z)}{2.5.8\ (z) \times 1.\ 7.13\ (z)} \times \frac{3^{z}}{2z}$.

By which equations it appears, that $\frac{F}{H} = \frac{F}{B}$ is = the limit of $\frac{1.3.\ 3.\ 5.\ 5.\ 7\ (2z)}{5.7.11.13.17.19\ (2z)} \times 3^{2z}$; which by Art. III. is $= \frac{4 \times \text{sine of } 90^\circ}{12 \times \text{sine of } 30^\circ}$ $= \frac{2}{3}$.

Therefore F is $= \frac{2B}{3}$.

It appears likewise, that

$\frac{G}{B}$ is = the limit of $\frac{1.3.5\ (z)\ \times\ 8.14.20\ (z)}{2.5.8\ (z)\ \times\ 7.13.19\ (z)} \times \frac{3^{z}}{2^{z}}$

= the limit of $\frac{1.3.5\ (z)\ \times\ 4.7.10\ (z)}{4.7.10\ (2z)} \times 6^{z}$

= the limit of $\frac{1.3.5\ (z)}{3z + 4.3z + 7.3z + 10\ (z)} \times 6^{z}$;

which, by Art. IV. is $= 2^{-\frac{1}{3}}$.

Therefore G is $= \frac{B}{2^{\frac{1}{3}}}$.

It also appears, that

I is = the limit of $\frac{1^2.\ 2^2.\ 3^2\ (z)}{2.3.5.7.8.10\ (2z)} \times \frac{3^{2z+1}}{2}$; which, by Art. III. $= \frac{2S}{4 \times \text{sine of } 60^\circ} = \frac{S}{3^{\frac{1}{2}}} = \frac{2C}{3^{\frac{1}{2}}}$.

Moreover it appears, that

$\frac{B \times K}{I}$ is $=$ the limit of $\frac{1.4.7 \ (z) \times 5.11.17 \ (z)}{2.5.8 \ (z) \times 1.3.5 \ (z)} \times \frac{1}{3^{z} - 1}$

$=$ the limit of $\frac{3z+2 \cdot 3z+5 \cdot 3z+8 \ (z)}{1 \cdot 3 \cdot 5 \ (z)} \times \frac{1}{2^{z} \cdot 3^{z-1}}$; which, by Art. IV. is $= \frac{3}{2^{\frac{1}{3}}}$.

$$\text{Therefore K is} = \frac{3 J}{2^{\frac{1}{3}} B} = \frac{2^{\frac{2}{3}} \cdot 3^{\frac{1}{2}} C}{B}.$$

And, in like manner, may a great number of other areas be compared.

Note. All the *whole* areas above-mentioned are ſuppoſed to begin where x begins, and to ſtand upon a baſe $= 1$.

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